L22 March 8 Compact

Tuesday, March 3, 2015 9:58 AM

Qu. $\mathbb{R}^n$ is not compact, but it has a nice property close to compact. What is it?

Locally Compact A topological space $(X, \mathcal{T})$ is locally compact if $\forall\, x \in X$ $\exists$ compact K such that $x \in \underbrace{\mathring{K} \subset K}_{\text{compact neighborhood}}$

Danger. The definition is inconsistent with others

Usually, for a topological property ☼D★,

X is locally ☼D★ if $\forall\, x \in X$ $\exists$ a local base of ☼D★-nbhds at x. That is, $\forall$ nbhd U of x $\exists$ ☼D★-nbhds V such that $x \in V \subset U$

Fact

$$\begin{array}{ccc} & & \text{locally compact} \\ & \nearrow & \uparrow \quad \Downarrow T_2 \quad \text{Why?} \\ \text{compact} & & \exists \text{ compact nbhd base} \end{array}$$

One-point Compactification

Given a locally compact T_2 space $(X, \mathcal{T})$

Then $\exists$ compact T_2 space $(X^*, \mathcal{T}^*)$ such that

(i) $X^* \setminus X$ is a singleton

(ii) $\mathcal{T} = \mathcal{T}^*|_X$

(iii) X is noncompact $\Rightarrow$ $\overline{X} = X^*$

X is compact $\Rightarrow$ $X^* \setminus X$ is isolated

Assume $\infty \notin X$, define $X^* = X \cup \{\infty\}$. and

$$\mathcal{J}^* = \mathcal{J} \cup \{ \{\infty\} \cup \underbrace{(X \setminus K)}_{\text{open as } X \text{ is } T_2} : K \subset X \text{ is compact} \}$$

① Verify that $\mathcal{J}^*$ is a topology

Crucial: $\bigcup_{\alpha \in I} (X \setminus K_\alpha) = X \setminus \bigcap_{\alpha \in I} K_\alpha$ both compact

$\bigcap_{j=1}^{n} (X \setminus K_j) = X \setminus \bigcup_{j=1}^{n} K_j$

② $(X^*, \mathcal{J}^*)$ is Hausdorff

The Key step: $x \in X$, $\infty \in X^*$

$x \in U$, $\infty \in \{\infty\} \cup (X \setminus K)$ and

$U \cap (X \setminus K) = \emptyset \iff x \in U \subset K$

③ $(X^*, \mathcal{J}^*)$ is compact

Key idea: If $X^* = \bigcup_{\alpha \in I} U_\alpha \cup \{\infty\} \cup (X \setminus K)$ then

$\{U_\alpha\}$ covers K and has a finite subcover

④ If X is compact, $\{\infty\} \cup (X \setminus X) \in \mathcal{J}^*$

$\therefore$ $\infty \in \{\infty\}$ is isolated

If $\overline{X} \subsetneqq X^*$, then $\overline{X} = X$

$\exists$ nbhd of ∞, $\{\infty\} \cup (X \setminus K)$ disjoint from X

Only possible $\{\infty\} = \{\infty\} \cup (X \setminus K)$, $K = X$

$\therefore$ X is compact

Heine-Borel Every open cover has a finite subcover

Other related compactness

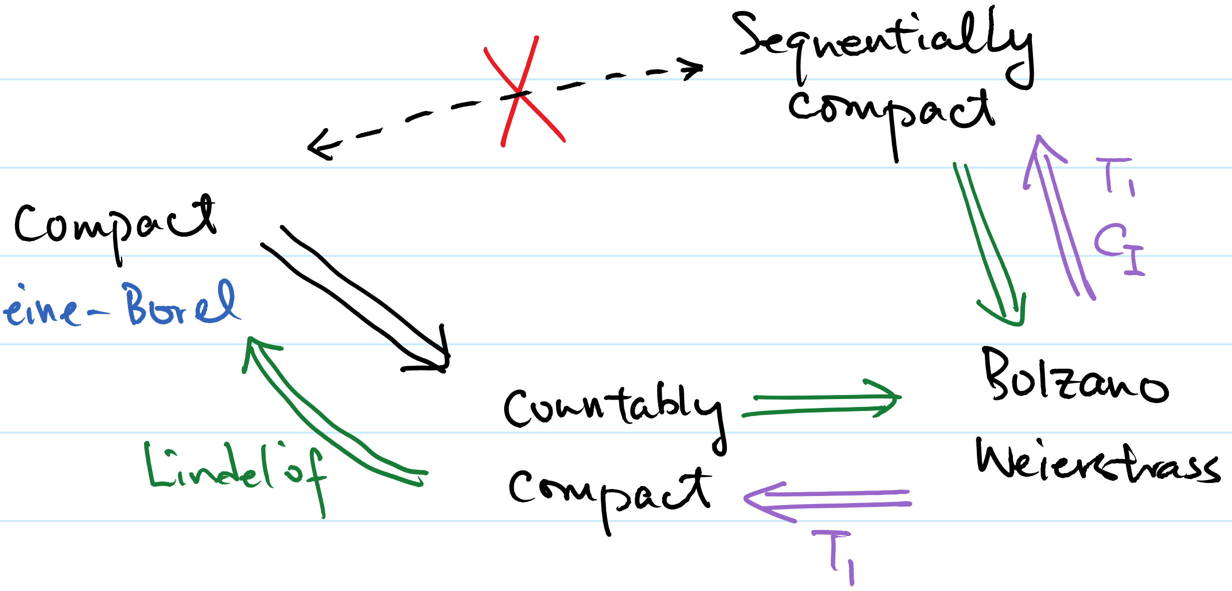
Countably Compact Every countable open cover has a finite subcover

Bolzano-Weierstrass Every infinite set has a cluster point in X.

If $A \subset X$ is infinite then $\exists\, x \in X$ s.t. $\forall\, U \in \mathcal{T}$ with $x \in U$, $U \cap A \setminus \{x\} \neq \emptyset$

Sequentially compact Every sequence has a convergent subsequence.

$\forall$ sequence (x_n) in X, $\exists$ subsequence (x_{n_k}) such that $x_{n_k} \longrightarrow x \in X$.



Sequentially Compact $\Longrightarrow$ Bolzano Weierstrass

Let $A \subset X$ be infinite

Create an infinite sequence $(a_n)_{n\in\mathbb{N}}$ in A

Get a convergent subsequence $a_{n_k} \longrightarrow x \in X$

Expect that $x \in A'$

Let $U \in \mathcal{J}$ with $x \in U$

By $a_{n_k} \to x$, $\exists\, k_0 \in \mathbb{N}$ s.t. $\forall\, k \geq k_0$

$a_{n_k} \in U$

$\therefore$ $a_{n_k} \in U \cap A$

How & why $a_{n_k} \in U \cap A \setminus \{x\}$

Method. Pick a distinct sequence

$a_n \in A$, i.e., $a_m \neq a_n$

$\therefore$ The set $\{a_n : n \in \mathbb{N}\}$ is infinite

$\exists\, a_{n_k} \longrightarrow x \in X$,

If $x = a_{n_l}$ for some $l \in \mathbb{N}$

then remove a_{n_l}

We have a subsequence $(a_{n_k})_{k\in\mathbb{N}}$

such that * $a_{n_k} \longrightarrow x$ as $k \to \infty$

* $a_{n_k} \neq a_{n_j}$ $\forall\, k, j \in \mathbb{N}$

* $a_{n_k} \neq x$ $\forall\, k \in \mathbb{N}$

Bolzano-Weierstrass $\xRightarrow[T_1]{C_I}$ Sequentially compact

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X

Consider $A = \{x_n : n \in \mathbb{N}\}$ Is it infinite?

If A is finite, $\exists$ constant subsequence and it converges

Assume that A is infinite, by Bolzano-Weierstrass, $\exists\ x \in A'$, i.e.
$\forall\ U \in \mathcal{J}$ with $x \in U$, $\phi \neq U \cap A \setminus \{x\}$

As X is C_I, let $\mathcal{U} = \{U_k : k \in \mathbb{N}\}$ be a local base at x. Then $U_k \cap A \setminus \{x\} \neq \phi$

Qu. How to pick a subsequence in $U_k \cap A$ and make sure it converges?

* First, since $A = \{x_n : n \in \mathbb{N}\}$ is infinite, so is the set $A \setminus \{x\}$

  We may assume $x_n \neq x$ and $x_m \neq x_n\ \forall m, n$
* Pick $x_{n_1} \in U_1 \cap A \setminus \{x\}$
* Consider $V_2 = U_1 \cap U_2 \setminus \{x_1, x_2, \cdots, x_{n_1}\} \in \mathcal{J}$ because X is T_1.

  $\exists\ x_{n_2} \in V_2 \cap A \setminus \{x\}$
* Similarly, $x_{n_k} \in V_k \cap A \setminus \{x\}$ where
$V_k = U_1 \cap \cdots \cap U_k \setminus \{x_1, x_2, \cdots, x_{n_{k-1}}\}$

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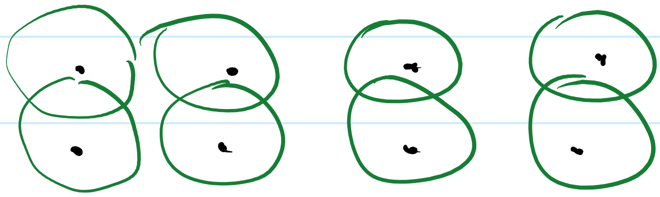
Countably Compact $\Longrightarrow$ Bolzano-Weierstrass

Qu. Think of an infinite set without cluster point

The obviously answer is $\mathbb{Z} \subset \mathbb{R}$ or $\mathbb{Z}^2 \subset \mathbb{R}^2$.

Qu. Find a countable cover for $\mathbb{R}^2$ which has no finite subcover

e.g. $\{B(x, \frac{1}{2}) : x \in \mathbb{Z}^2\} \cup \{\mathbb{R}^2 \setminus \mathbb{Z}^2\}$



Qu. Produce a proof for the contrapositive from this example.

Let $A \subset X$ be an infinite set and $A' = \emptyset$

Take a countable subset $B \subset A$, $B' = \emptyset$

① B is discrete

Let $b \in B$. As $B' = \emptyset$, $b \notin B'$

$\therefore \exists\, U_b \in \mathcal{J}$ with $b \in U_b$ such that

$U_b \cap B \setminus \{b\} = \emptyset$, i.e., $U_b \cap B = \{b\}$

② $X \setminus B$ is open

Let $x \in X \setminus B$. As $x \notin B'$

$\exists\, U_x \in \mathcal{J}$ with $x \in U_x$, $U_x \cap B \setminus \{x\} = \emptyset$

$\therefore U_x \setminus \{x\} \subset X \setminus B$

$x \in U_x \subset X \setminus B$

Bolzano-Weierstrass $\xRightarrow{T_1}$ Countably Compact

Let $\{U_n : n\in\mathbb{N}\}$ be a countable open cover for X

How to get an infinite set?

Idea, if $U_1\cup\cdots\cup U_n \not\supset X$ then $\exists\, x_n$ beyond!

But if $x_n \xrightarrow{\text{cluster}}$ at x, then many points can be covered by a single U_n

Take any $x_1 \in \bigcup_{n=1}^{\infty} U_n = X$

Then $x_1\in U_{n_1}$ but $x_1\notin U_1\cup\cdots\cup U_{n_1-1}$

If $X\neq \bigcup_{n=1}^{n_1} U_n$ then we have

$$x_2\in U_{n_2}, \quad x_2\notin U_1\cup\cdots\cup U_{n_1}\cup\cdots\cup U_{n_2-1}$$

Claim. The process must stop at finite step!

i.e. $X=\bigcup_{n=1}^{n_k} U_n$, finite subcover

Assume not, $\exists$ infinite $\{x_q : q\in\mathbb{N}\}$ where

$x_q\in U_{n_q}$ but $x_q\notin U_j$ for $j<n_q$

We will show $A'=\emptyset$, i.e., $\forall\, x\in X$, $x\notin A'$

Since $X=\bigcup_{n=1}^{\infty} U_n$, $x\in U_m$ for some m.

Thus, we have $\qquad n_N < m \le n_{N+1}$

$$\underbrace{\begin{array}{ccccccccccccc} U_1 & U_2 & \cdots & U_{n_1} & \cdots & U_{n_2} & \cdots & U_{n_N} & \cdots & U_m & \cdots & U_{n_{N+1}} & \cdots \\ & & & \cup\!\!\mid & & \cup\!\!\mid & & \cup\!\!\mid & & \cup\!\!\mid & & \cup\!\!\mid & \\ & & & x_1 & & x_2 & & x_N & & x & & x_{N+1} & \end{array}}_{\text{By construction, no } x_{N+1},\, x_{N+2},\, x_{N+3},\, \dots}$$

X is T_1, $\therefore\ x\in V = U_m\setminus\{x_1,x_2,\dots,x_N \text{ or } x_{N+1}\}\in\mathcal{T}$

But $\quad V\cap A\setminus\{x\}=\emptyset$